STA502: Math & Stat for MBA
Problem Set 4

Question 1. Let $gGDP_t$ denote the annual percentage change in gross domestic product and let int_t denote a short-term interest rate. Suppose that $gGDP_t$ is related to interest rate by

$$gGDP_t = \alpha_0 + \delta_0 int_t + \delta_1 int_{t-1} + u_t$$

where u_t is uncorrelated with int_t, int_{t-1}, and all other past values of interest rates.

Suppose that the Federal Reserve follows the policy rule:

$$int_t = \gamma_0 + \gamma_1(gGDP_{t-1} - 3) + v_t,$$

where $\gamma_1 > 0$. In behavioral terms, $\gamma_1 > 0$ means that when last year's GDP growth is above 3%, the Fed increases interest rates to prevent an "overheated" economy.

(a) If v_t is uncorrelated with all past values of int_t and u_t, argue that int_t must be correlated with u_{t-1}. (Hint: Lag the first equation for one time period and substitute for $gGDP_{t-1}$ in the second equation.) We want to show that $Cov(int_t, u_{t-1}) \neq 0$

(b) Which Gauss-Markov assumption does $Corr(int_t, u_{t-1})$ violate?

Solution.

1. Lag the first equation : $gGDP_{t-1} = \alpha_0 + \delta_0 int_{t-1} + \delta_1 int_{t-2} + u_{t-1}$. Substitute for $gGDP_{t-1}$ in the second equation, we get $int_t = (\gamma_0 - 3\gamma_1 + \alpha\gamma_1) + \gamma_1\delta_0 int_{t-1} + \gamma_1\delta_1 int_{t-2} + (\gamma_1 u_{t-1} + v_t)$, which is the AR(2) model. (We assume the stationarity in the problem ; beyond the scope)
   By assumption, u_t is uncorrelated with $int_t \sim int_1$ and v_t is uncorrelated with $int_t \sim int_1$, $u_t \sim u_1$.
   Since int_t contains $(\gamma_1 u_{t-1} + v_t)$, $cov(int_t, u_{t-1}) = \gamma_1 var(u_{t-1}) \neq 0$. Hence int_t is correlated with u_{t-1}.
2. Since int_t is correlated with u_{t-1}, A3Rmi collapsed. But A3Rsru still holds by the assumption. Based on the assumption A1, A2, if A3Rmi condition additionally holds, OLS yields unbiased & consistent estimator. But since A3 condition relaxed to A3Rsru, only consistency of estimator guaranteed. In general, lag variable model doesn't provide A3Rmi condition.

Question 2. Consider the stationary AR(1) model

$$y_t = \beta_1 y_{t-1} + \epsilon_t \ with \ |\beta_1| < 1$$

where ϵ_t is i.i.d.(0,σ^2) and ϵ_t is independent $y_{t-1}, y_{t-2}, ...$

(a) Derive the OLS estimator of β_1, $\hat{\beta}_1$

(b) Is $\hat{\beta}_1$ an unbiased estimator for β_1? Is it BLUE? Clearly explain your answers.

(c) Is $\hat{\beta}_1$ an consistent estimator for β_1? (Not required. TA should cover intuition with respect to BigData)

Solution.

1. $\hat{\beta}_1 = (y'_{t-1}y_{t-1})^{-1}y'_{t-1}y_t = \frac{\sum_{t=1}^{t}(y_{t-1} - \bar{y_{t-1}})y_t}{\sum_{t=1}^{t}(y_{t-1} - \bar{y_{t-1}})^2}$. Assumption for the initial value, y_0 is needed.
2. $E(\hat{\beta}_1) = E((y'_{t-1}y_{t-1})^{-1}y'_{t-1}y_t) = E((y'_{t-1}y_{t-1})^{-1}y'_{t-1}(\beta_1 y_{t-1} + \epsilon_t))$.
   $= E((y'_{t-1}y_{t-1})^{-1}y'_{t-1}\beta_1 y_{t-1}) + E((y'_{t-1}y_{t-1})^{-1}y'_{t-1}\epsilon_t)$ (by A3Rmi)
   $= \beta_1 + (y'_{t-1}y_{t-1})^{-1}y'_{t-1}E(\epsilon_t)$
   $= \beta_1$ (by A2)

   Unbiasedness holds when the condition A3Rmi or stronger. But in the lag variable model, y_{t-1} contains ϵ_{t-1}, A3Rmi cannot be satisfied. $\hat{\beta}_{1OLS}$ is not BLUE.

3. $\text{plim}(\hat{\beta}_1 - \beta_1) = (y'_{t-1}y_{t-1})^{-1}y'_{t-1}\epsilon_t = (\frac{(y'_{t-1}y_{t-1})^{-1}}{N})(\frac{(y'_{t-1}\epsilon_t)}{N})$, and if sample goes to infinity, $\frac{(y'_{t-1}y_{t-1})^{-1}}{N}$ converges to $\Sigma^{-1}_{y_{t-1}}$ and $\frac{(y'_{t-1}\epsilon_t)}{N}$ goes to 0 if A3Rsru is satisfied. Since ϵ_t is iid and independent with $y_{t-1} \sim y_1$, $cov(y_{t-1}, \epsilon_t) = 0$. A3Rsru holds. Multiplying the two limit, it goes to 0.
If estimator is consistent, estimator converges to true parameter if sample size is large.