1. $y_t = \beta_1 + \beta_2 x_t + \varepsilon_t \quad (t=1,2,\cdots,T) \Rightarrow y = X\beta + \varepsilon$

(1) $X = [\mathbb{1} \ x] \quad \mathbb{1} = (1,1,\cdots,1)^T, \quad x = (x_1, x_2, \cdots, x_T)^T$

$$X'X = \begin{pmatrix} \mathbb{1}^T \\ x^T \end{pmatrix} (\mathbb{1} \ x) = \begin{pmatrix} \mathbb{1}^T\mathbb{1} & \mathbb{1}^T x \\ x^T\mathbb{1} & x^T x \end{pmatrix} = \begin{pmatrix} T & \sum_t x_t \\ \sum_t x_t & \sum_t x_t^2 \end{pmatrix}$$

$$X'y = \begin{pmatrix} \mathbb{1}^T \\ x^T \end{pmatrix} y = \begin{pmatrix} \mathbb{1}^T y \\ x^T y \end{pmatrix} = \begin{pmatrix} \sum_t y_t \\ \sum_t x_t y_t \end{pmatrix}$$

$$(X'X)^{-1} = \frac{1}{T\sum_t x_t^2 - \left(\sum_t x_t\right)^2} \begin{pmatrix} \sum_t x_t^2 & -\sum_t x_t \\ -\sum_t x_t & T \end{pmatrix} = \frac{1}{T\sum_t (x_t - \bar{x})^2} \begin{pmatrix} \sum_t x_t^2 & -\sum_t x_t \\ -\sum_t x_t & T \end{pmatrix}$$

$$\hat{\beta} = (X'X)^{-1}X'y = \frac{1}{T\sum_t x_t^2 - \left(\sum_t x_t\right)^2} \begin{pmatrix} \sum_t x_t^2 & -\sum_t x_t \\ -\sum_t x_t & T \end{pmatrix} \begin{pmatrix} \sum_t y_t \\ \sum_t x_t y_t \end{pmatrix}$$

$$\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{pmatrix} \dfrac{\sum_t x_t^2 \sum_t y_t - \sum_t x_t \sum_t x_t y_t}{T\sum_t (x_t - \bar{x})^2} \\[2ex] \dfrac{-\sum_t x_t \sum_t y_t + T\sum_t x_t y_t}{T\sum_t (x_t - \bar{x})^2} \end{pmatrix}$$

$$\hat{\beta}_1 = \frac{\sum_t x_t^2 \sum_t y_t - \sum_t x_t \sum_t x_t y_t}{T\sum_t (x_t - \bar{x})^2} = \frac{\bar{y}\sum_t x_t^2 - \bar{x}\sum_t x_t y_t}{\sum_t (x_t - \bar{x})^2} = \frac{\bar{y}\sum_t (x_t - \bar{x})^2 + \bar{y}T\bar{x}^2 - \bar{x}\sum_t x_t y_t}{\sum_t (x_t - \bar{x})^2}$$

$$= \bar{y} + \frac{\sum_t \bar{x}^2 y_t - \bar{x}\sum_t x_t y_t}{\sum_t (x_t - \bar{x})^2} = \bar{y} - \frac{\bar{x}\sum_t (x_t - \bar{x}) y_t}{\sum_t (x_t - \bar{x})^2} = \bar{y} - \bar{x}\hat{\beta}_2$$

$$\hat{\beta}_2 = \frac{-\sum_t x_t \sum_t y_t + T\sum_t x_t y_t}{T\sum_t (x_t - \bar{x})^2} = \frac{T\sum_t x_t y_t - T\bar{x}\sum_t y_t}{T\sum_t (x_t - \bar{x})^2} = \frac{\sum_t (x_t - \bar{x}) y_t}{\sum_t (x_t - \bar{x})^2}$$

(2) $Var(\hat{\beta}) = \sigma_e^2 (X'X)^{-1} = \dfrac{\sigma_e^2}{T\sum_t (x_t - \bar{x})^2} \begin{pmatrix} \sum_t x_t^2 & -\sum_t x_t \\ -\sum_t x_t & T \end{pmatrix} = \dfrac{\sigma_e^2}{T\sum_t (x_t - \bar{x})^2} \begin{pmatrix} \sum_t x_t^2 & -T\bar{x} \\ -T\bar{x} & T \end{pmatrix}$

On the other hand, $Var(\hat{\beta}) = \begin{pmatrix} Var(\hat{\beta}_1) & Cov(\hat{\beta}_1, \hat{\beta}_2) \\ Cov(\hat{\beta}_2, \hat{\beta}_1) & Var(\hat{\beta}_2) \end{pmatrix}$

$$\therefore Var(\hat{\beta}_1) = \frac{\sigma_e^2 \sum_t x_t^2}{T\sum_t (x_t - \bar{x})^2} = \sigma_e^2 \times \frac{\sum_t (x_t - \bar{x})^2 + T\bar{x}^2}{T\sum_t (x_t - \bar{x})^2}$$

$$= \sigma_e^2 \left(\frac{1}{T} + \frac{\bar{x}^2}{\sum_t (x_t - \bar{x})^2} \right)$$

$$Var(\hat{\beta}_2) = \frac{\sigma_e^2}{\sum_t (x_t - \bar{x})^2}$$

$$Cov(\hat{\beta}_1, \hat{\beta}_2) = \frac{-\sigma_e^2 \bar{x}}{\sum_t (x_t - \bar{x})^2}$$

2. $n=145$, $\log(cost) = \beta_1 + \beta_2 \log(Output) + \beta_3 \log(Price\ of\ Labor) + \beta_4 \log(Cost\ of\ Capital) + \beta_5 \log(Price\ of\ Fuel) + \varepsilon$

(1) $H_0: \beta_2 = 1$ (It means if output increases by 1%, cost also increases by 1%, which is constant returns to scale)

$H_A: \beta_2 < 1$ (It means if output increases by 1%, cost increases by less than 1%, which is increasing returns to scale)

we have to do one-sided t-test.

$$T = \frac{\hat{\beta}_2 - \beta_2}{s.e(\hat{\beta}_2)} = \frac{0.720 - 1}{0.0175} = -16 \sim t(n-5) = t(140)$$

Since the critical value at the 5% significance level for t(140) distribution is −1.655, we can reject the null hypothesis, Ho.

(2) The coefficient of each independent variable is the power. Homogeneous of degree one in prices means the sum of power of each price variable (Price of Labor, Cost of Capital, Price of fuel) should be one.

Now, we can define Ho & HA and do two-sided t-test

$H_0: \beta_3 + \beta_4 + \beta_5 = 1$, $H_A: \beta_3 + \beta_4 + \beta_5 \neq 1$

Critical value at the 5% significance level for t(140) distribution is 1.977

$$T = \frac{(\hat{\beta}_3 + \hat{\beta}_4 + \hat{\beta}_5) - 1}{s.e(\hat{\beta}_3 + \hat{\beta}_4 + \hat{\beta}_5)} = \frac{(\hat{\beta}_3 + \hat{\beta}_4 + \hat{\beta}_5) - 1}{\sqrt{Var(\hat{\beta}_3) + Var(\hat{\beta}_4) + Var(\hat{\beta}_5) + 2(Cov(\hat{\beta}_3, \hat{\beta}_4) + Cov(\hat{\beta}_3, \hat{\beta}_5) + Cov(\hat{\beta}_4, \hat{\beta}_5))}}$$

$$= \frac{(0.436 + 0.220 + 0.427) - 1}{\sqrt{0.0847 + 0.115 + 0.0101 + 2(0.0237 - 0.0109 - 0.00663)}} = 0.1961$$

Therefore, we cannot reject the null hypothesis, Ho.

3. $X = [x_1 \ x_2]$, $y \sim N(X\beta, \sigma_u^2 I_n)$, $n = 11$ $\quad \hat{\sigma}_u^2 = \frac{1}{9}\sum_i (y_i - x_{1i})^2 = \frac{1}{9} \times \frac{1}{3} = \frac{1}{27}$

$$\hat{\beta} = \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = (X'X)^{-1}X'y = \left[\begin{pmatrix} x_1' \\ x_2' \end{pmatrix}(x_1 \ x_2)\right]^{-1}\begin{pmatrix} x_1'y \\ x_2'y \end{pmatrix}$$

$$= \begin{pmatrix} x_1'x_1 & x_1'x_2 \\ x_2'x_1 & x_2'x_2 \end{pmatrix}^{-1}\begin{pmatrix} x_1'y \\ x_2'y \end{pmatrix} = \frac{1}{x_1'x_1x_2'x_2 - x_1'x_2x_2'x_1}\begin{pmatrix} x_2'x_2 & -x_1'x_2 \\ -x_2'x_1 & x_1'x_1 \end{pmatrix}\begin{pmatrix} x_1'y \\ x_2'y \end{pmatrix}$$

$$= \frac{1}{4-1}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \hat{\beta}_1 = 1, \ \hat{\beta}_2 = 0$$

(1) $\hat{y}_z = z\hat{\beta} = z(X'X)^{-1}X'y \sim N(\underset{\overset{\|}{y_z}}{z\beta}, \sigma_u^2 z(X'X)^{-1}z')$

$$Var(\hat{y}_z) = z(X'X)^{-1}X' Var(y) X (X'X)^{-1}z' = z(X'X)^{-1}X'\sigma_u^2 I_n X (X'X)^{-1}z'$$

$$= \sigma_u^2 z(X'X)^{-1}z'$$

$$y_z \sim N(z\beta, \sigma_u^2)$$

$\hat{y}_z - y_z \sim N(0, \sigma_u^2(1 + z(X'X)^{-1}z'))$ ($\because$ $\hat{y}_z$ and y_z are Independent)

$$\frac{\hat{y}_z - y_z}{\sqrt{\sigma_u^2(1+z(X'X)^{-1}z')}} \sim N(0,1),$$

$$\frac{\hat{y}_z - y_z}{\sqrt{\hat{\sigma}_u^2(1+z(X'X)^{-1}z')}} = \frac{\hat{y}_z - y_z}{\sqrt{\sigma_u^2(1+z(X'X)^{-1}z')}} \Bigg/ \sqrt{\frac{\hat{\sigma}_u^2(n-2)}{\sigma_u^2}\Big/(n-2)} \sim t(n-2)$$

since $(\hat{y}_z, y_z)$ and $\hat{\sigma}_u^2$ are independent.

PI will be $\hat{y}_z \pm t_{0.1}(n-2) \cdot \sqrt{\hat{\sigma}_u^2(1+z(X'X)^{-1}z')}$, $t_{0.1}(9) = 1.383$

For observation 12, $z = (5 \ -2)$, $\hat{y}_z = z\hat{\beta} = (5 \ -2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 5$

PI: $5 \pm 1.383\sqrt{\hat{\sigma}_u^2(1+26)} = 5 \pm 1.383\sqrt{27\hat{\sigma}_u^2} = (3.617, 6.383)$

For observation 13, $z = (3 \ -7)$, $\hat{y}_z = z\hat{\beta} = (3 \ -7)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3$

PI: $3 \pm 1.383\sqrt{\hat{\sigma}_u^2(1+\frac{152}{3})} \simeq (1.0868, 4.9131)$

(2) Let expected value of y_{12} and y_{13} be μ_{12} and μ_{13}

First, look at $\hat{\mu}_z$, where z is the new point.

$$\hat{\mu}_z \sim N\left(\mu_z,\ \sigma_\varepsilon^2 z(X'X)^{-1}z'\right) = N\left(z\beta,\ \sigma_\varepsilon^2 z(X'X)^{-1}z'\right)$$

$$\left(\begin{array}{l} \hat{\mu}_z = \hat{\beta}_1 z_1 + \hat{\beta}_2 z_2 = z\hat{\beta} = z(X'X)^{-1}X'y \\ Var(\hat{\mu}_z) = z(X'X)^{-1}X'\,Var(y)\,X(X'X)^{-1}z' = \sigma_\varepsilon^2 z(X'X)^{-1}z' \end{array}\right)$$

$$\frac{\hat{\mu}_z - \mu_z}{\sqrt{\sigma_\varepsilon^2 z(X'X)^{-1}z'}} \sim N(0,1) \Rightarrow \frac{\hat{\mu}_z - \mu_z}{\sqrt{\hat{\sigma}_\varepsilon^2 z(X'X)^{-1}z'}} \sim t(n-2) = t(9)$$

($\because$ $\hat{\mu}_z$ and $\hat{\sigma}_\varepsilon^2$ are independent)

PI for μ_z: $\hat{\mu}_z \pm t_{0.1}(n-2)\sqrt{\hat{\sigma}_\varepsilon^2 z(X'X)^{-1}z'}$

For observation 12, $z = (5\ -2)$, $\hat{\mu}_z = z\hat{\beta} = (5\ -2)\begin{pmatrix}1\\0\end{pmatrix} = 5$

$z(X'X)^{-1}z' = (5\ -2)\begin{pmatrix}\frac{2}{3} & -\frac{1}{3}\\ -\frac{1}{3} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}5\\-2\end{pmatrix} = 26$

PI for $\mu_{12} = 5 \pm 1.383\sqrt{26\,\hat{\sigma}_\varepsilon^2} = (3.6428, 6.3571)$

For observation 13, $z = (3\ -7)$, $\hat{\mu}_z = z\hat{\beta} = (3\ -7)\begin{pmatrix}1\\0\end{pmatrix} = 3$

$z(X'X)^{-1}z' = (3\ -7)\begin{pmatrix}\frac{2}{3} & -\frac{1}{3}\\ -\frac{1}{3} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}3\\-7\end{pmatrix} = \frac{152}{3}$

PI for $\mu_{13} = 3 \pm 1.383\sqrt{\frac{152}{3}\hat{\sigma}_\varepsilon^2} = (1.1054, 4.8945)$

(3) Prediction intervals for y for observations 12 & 13 are longer than those for expected value of y_{12} and y_{13}. That's because we should consider the noise term ε. $\hat{y}_z$ has more variation than $\hat{\mu}_z$ due to ε