Q1. $y_t = \beta_1 + \beta_2 x_t + \varepsilon_t$, $t = 1, 2, \cdots T$

Q1-1. Obtain formulae for $\hat{\beta}_1$, $\hat{\beta}_2$

Answer $X = [i \ x] \cdots$

$$X'X = \begin{bmatrix} i \\ x \end{bmatrix} [i \ x] = \begin{bmatrix} i'i & i'x \\ x'i & x'x \end{bmatrix}$$

$$= \begin{bmatrix} T & \sum_t x_t \\ \sum_t x_t & \sum_t x_t^2 \end{bmatrix} \cdots ①$$

$$X'Y = \begin{bmatrix} i \\ x \end{bmatrix}' \cdot y = \begin{bmatrix} \sum_t y_t \\ \sum_t x_t y_t \end{bmatrix}' \cdots ②$$

$$(X'X)^{-1} = \frac{1}{T \cdot \sum_t x_t^2 - (\sum_t x_t)^2} \begin{bmatrix} \sum_t x_t^2 & -\sum_t x_t \\ -\sum_t x_t & T \end{bmatrix}$$

$$= \frac{1}{T \cdot \sum_t (x_t - \bar{x})^2} \begin{bmatrix} \sum_t x_t^2 & -\sum_t x_t \\ -\sum_t x_t & T \end{bmatrix} \cdots ③$$

$$\hat{\beta} = (X'X)^{-1} \cdot (X'Y) = ③ \times ②$$

$$= \begin{bmatrix} \sum_t x_t^2 & -\sum_t x_t \\ -\sum_t x_t & T \end{bmatrix} \begin{bmatrix} \sum_t y_t \\ \sum_t x_t y_t \end{bmatrix} \times \frac{1}{T \sum (x_t - \bar{x})^2}$$

$$\hat{\beta} = \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{bmatrix} \dfrac{\sum_t x_t^2 \cdot \sum_t y_t - \sum_t x_t \cdot \sum_t x_t y_t}{T \sum (x_t - \bar{x})^2} \\ \\ \dfrac{-\sum_t x_t \sum y_t + T \cdot \sum_t x_t \cdot y_t}{T \sum (x_t - \bar{x})^2} \end{bmatrix}$$

(1) $\hat{\beta}_1$

$$\hat{\beta}_1 = \frac{\sum_t x_t^2 \sum y_t - \sum_t x_t \sum_t x_t y_t}{T \sum (x_t - \bar{x})^2}$$

$\Downarrow$

Devide the denominator & numerator by T

$\Downarrow$

$$\frac{\bar{y} \cdot \sum_t x_t^2 - \bar{x} \cdot \sum_t x_t y_t \cdots ⓐ}{\sum (x_t - \bar{x})^2 \cdots ⓑ}$$

ⓐ $\bar{y} \cdot \sum_t x_t^2 - \bar{x} \cdot \sum_t x_t y_t$

$= \underline{\bar{y} \cdot \sum_t x_t^2 - \bar{x}^2 \cdot \sum_t y_t} - \underline{\bar{x} \cdot \sum_t x_t y_t + \bar{x}^2 \cdot \sum y_t}$

$= \bar{y} \{ \sum_t x_t^2 - \frac{1}{T} (\sum x_t)^2 \} - \bar{x} \cdot \sum x_t y_t + \bar{x}^2 \cdot \sum y_t$

$= \bar{y} (\sum (x_t - \bar{x})^2 - \bar{x} \{ \sum x_t y_t - \frac{1}{T} \sum x_t y_t \}$

$= \bar{y} \sum (x_t - \bar{x})^2 - T \cdot \bar{x} (\frac{1}{T} \sum x_t y_t - \frac{1}{T^2} \sum x_t y_t)$

$= \bar{y} \sum (x_t - \bar{x})^2 - T \cdot \bar{x} (\frac{1}{T} \sum x_t y_t - \bar{x} \bar{y})$

$= \bar{y} \sum (x_t - \bar{x})^2 - \bar{x} \cdot \sum x_t y_t + T \cdot \bar{x}^2 \cdot \bar{y}$ //

$$\therefore \hat{\beta}_1 = \frac{\bar{y} \sum (x_t - \bar{x})^2 - \bar{x} \sum x_t y_t + T \bar{x}^2 \bar{y}}{\sum (x_t - \bar{x})^2}$$

$$= \bar{y} - \left(\frac{\bar{x} \sum x_t y_t - \bar{x}^2 \sum y_t}{\sum (x_t - \bar{x})^2} \right)$$

$$= \bar{y} - \left(\frac{\bar{x} \{ \sum (x_t - \bar{x}) y_t}{\sum (x_t - \bar{x})^2} \right) //$$

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(2) $\hat{\beta}_2$

$$\hat{\beta}_2 = \frac{-\sum_t x_t \sum_t y_t + T \cdot \sum_t x_t y_t}{T \cdot \sum_t (x_t - \bar{x})^2}$$

$\downarrow$ Devide the denominator & numerator by T

$$= \frac{-\frac{1}{T}\sum x_t \sum y_t + \sum x_t y_t}{\sum_t (x_t - \bar{x})^2}$$

$$= \frac{-\bar{x}\sum y_t + \sum x_t \cdot y_t}{\sum (x_t - \bar{x})^2}$$

$$\therefore \hat{\beta}_2 = \frac{\sum (x_t - \bar{x}) y_t}{\sum (x_t - \bar{x})^2} \quad //$$

Q2. $\log(\text{cost}) = \underset{(1.77)}{-3.53} + \underset{(0.0175)}{0.720} \log(\text{output})$

$+ \underset{(0.291)}{0.436} \log(\text{price of labor}) + \underset{(0.339)}{0.220} \log(\text{C.C})$

$+ \underset{(0.100)}{0.427} \log(\text{price of fuel}) + \hat{\varepsilon}$

Q2-1. hypothess test (about β_2), $\alpha = 0.05$, $n = 145$

Answer

$H_0 : \beta_2 = 1$

$H_1 : \beta_2 < 1$

$$T = t = \frac{\beta_2 - \hat{\beta}_2}{\sqrt{\hat{Var}(\hat{\beta}_2)}} \sim t^*_{0.05}(140)$$

$$= \frac{0.720 - 1}{0.0175} < t_{0.05}(140) = 1.6558 \text{v}$$

$$-16 < -1.65581)$$

$T < t^*$ ... we reject H_0

Q2-2.

invest → price of labor, cost of capital, price of fuel

⇒ price of labor + cost of capital + price of fuel

= 100% effective.

$\Downarrow$ hypothesis test

$H_0: \beta_3 + \beta_4 + \beta_5 = 1$

$H_1: \beta_3 + \beta_4 + \beta_5 \neq 1$

$\Downarrow$ Cobb-Douglas production Function

$\omega = \beta_3 + \beta_4 + \beta_5$

$\Downarrow$

$H_0: \omega = 1$, $H_1: \omega \neq 1$

① $E(\hat{\omega}) = E(\hat{\beta_3}) + E(\hat{\beta_4}) + E(\hat{\beta_5})$

$= \beta_3 + \beta_4 + \beta_5$

② $Var(\hat{\omega}) = Var(\hat{\beta_3}) + Var(\hat{\beta_4}) + Var(\hat{\beta_5})$
$+ 2cov(\hat{\beta_3}, \hat{\beta_4}) + 2cov(\hat{\beta_3}, \hat{\beta_5}) + 2cov(\hat{\beta_4}, \hat{\beta_5})$

$= (0.0847 + 0.115 + 0.0101) + 2(0.0237$
$- 0.0109 - 0.00663) = \sqrt{0.22214}$

③ $\tau = \dfrac{\hat{\omega} - (\beta_3 + \beta_4 + \beta_5)}{\sqrt{0.22214}} = \dfrac{1.083 - 1}{\sqrt{0.22214}}$

$\fallingdotseq 0.176102$

④ $t^*_{0.025}(140) \fallingdotseq 1.977054$

$\therefore \tau(0.176102) < t^*_{0.025}(140)$

$\longrightarrow$ H_0 is not rejected.

Q3. $y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + \varepsilon_i$, $i = 1, 2, \cdots 11.$

$$\boxed{\begin{array}{lll} x_1'x_1 = 2, & x_2'x_2 = 2 & x_1'x_2 = 1 \\ x_1'y = 2 & x_2'y = 1 & y'y = 4/3 \end{array}}$$

$\Downarrow$

① $\sum_i x_{1i}^2 = 2$ ② $\sum_i x_{1i} x_{2i} = 2$

③ $\sum_i x_{1i} y_i = 2$ ④ $\sum_i y_i^2 = 7/3$

⑤ $\sum_i x_{2i} y_i = 1$ ⑥ $\sum_i x_{2i}^2 = 2$

$\Downarrow$

ⓐ $X'X = \begin{bmatrix} \Sigma_{1i}^2 & \Sigma x_{1i} x_{2i} \\ \Sigma x_{1i} x_{2i} & \Sigma x_{2i}^2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

ⓑ $X'Y = \begin{bmatrix} \Sigma x_{1i} y \\ \Sigma x_{2i} y \end{bmatrix}' = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

ⓒ $\hat{\beta} = (X'X)^{-1} \cdot X'Y$

$= \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\hat{\beta_1} = 1$, $\hat{\beta_2} = 0$

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ⓓ $cov(\hat{\beta}) = \sigma^2 \cdot (X'X)^{-1}$

$= \frac{1}{3}\sigma^2 \cdot \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

ⓔ $\hat{\sigma}^2 = \frac{1}{n-2}\sum \hat{\varepsilon}_i^2$

$\hat{\beta}_1 = 1$ $\quad \hat{\beta}_2 = 0$

$= \frac{1}{n-2}\sum (y_i - \hat{\beta}_1 x_{1i} - \hat{\beta}_2 x_{2i})^2$

$= \frac{1}{n-2}\sum (y_i - x_{1i})^2$

$= \frac{1}{9}\left\{ \sum y_i^2 - 2 \cdot \sum x_{1i} y_i + \sum x_{1i}^2 \right\}$

$= \frac{1}{9}\left\{ \frac{7}{3} - (2\times 2) + 2 \right\}$

$= \frac{1}{9} \times \frac{1}{3} = \frac{1}{27}$

$\therefore \hat{\sigma}^2 = \frac{1}{27}$

$[\ \hat{\sigma} = \frac{1}{3\sqrt{3}}$

80% prediction interval

Q3-1. dependent variable y for obs. 12, 13

Answer $\quad y_i = x_{1i} + 0 \cdot x_{2i} + \varepsilon_i$

obs. 12 > $\quad 5 \quad -2$

obs. 13 > $\quad 3 \quad -7$

$y_{12} = 5 + \varepsilon_{12}$, $y_{13} = 3 + \varepsilon_{13}$

80% prediction interval

80% prediction interval > obs. 12 : $5 + \varepsilon_{12} \pm 1.28(0.1945)$

$[4.754 + \varepsilon_{12},\ 5.246 + \varepsilon_{12}]$

obs. 13 : $3 + \varepsilon_{13} \pm 1.28(0.1945)$

$[2.754 + \varepsilon_{13},\ 3.246 + \varepsilon_{13}]$

Q3-2. 80% prediction interval

expected value y_{12}, y_{13}

Answer $\quad \hat{y}_i = \hat{x}_{1i}$

$\hat{y}_{12} = \hat{x}_{1,12} = 5$

$\hat{y}_{13} = \hat{x}_{1,13} = 3$

80% prediction interval > $5 \pm 1.28(0.1945)$

y_{12} : $[4.754,\ 5.246]$

y_{13} : $[2.754,\ 3.246]$

Q3-3. Do the answers above differ? why?

Answer

The errors are reflected in answer Q3-1, because we don't know the actual value of y_{12} and y_{13}.