Math & Stat for MBA
Problem Set #1

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Ans 1-1 Where $y_t = \beta_1 + \beta_2 x_t + \epsilon_t$, $t = 1, 2, \cdots, T$

The least squares estimator $\hat{\beta} = (X'X)^{-1}X'y$

i) $X'X = \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_T \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_T \end{bmatrix} = \begin{bmatrix} T & \Sigma_t x_t \\ \Sigma_t x_t & \Sigma_t x_t^2 \end{bmatrix}$

$(X'X)^{-1} = \begin{bmatrix} \Sigma_t x_t^2 & -\Sigma_t x_t \\ -\Sigma_t x_t & T \end{bmatrix} / \Delta$, where $\Delta = T\Sigma_t x_t^2 - (\Sigma_t x_t)^2$
$= T\Sigma_t (x_t - \bar{x})^2$

ii) $X'y = \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_t \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_t \end{bmatrix} = \begin{bmatrix} \Sigma_t y_t \\ \Sigma_t x_t y_t \end{bmatrix}$

Then, we can carry out $\hat{\beta}$

$$\hat{\beta} = (X'X)^{-1}X'y = \begin{bmatrix} \Sigma_t x_t^2 & -\Sigma_t x_t \\ -\Sigma_t x_t & T \end{bmatrix} \begin{bmatrix} \Sigma_t y_t \\ \Sigma_t x_t y_t \end{bmatrix} \times \frac{1}{T\Sigma_t (x_t - \bar{x})^2}$$

$$= \begin{bmatrix} \Sigma_t x_t^2 \cdot \Sigma_t y_t - \Sigma_t x_t \cdot \Sigma_t x_t y_t \\ -\Sigma_t x_t \cdot \Sigma_t y_t + T\Sigma_t x_t y_t \end{bmatrix} \times \frac{1}{T\Sigma_t (x_t - \bar{x})^2} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix}$$

$$\therefore \hat{\beta}_1 = \bar{y} - \frac{\Sigma_t (x_t - \bar{x}) y_t}{\Sigma (x_t - \bar{x})^2} \bar{x} \qquad \hat{\beta}_2 = \frac{\Sigma_t (x_t - \bar{x}) y_t}{\Sigma (x_t - \bar{x})^2}$$

Ans 1-2

As we know, $Cov(\hat{\beta}) = \sigma^2 (X'X)^{-1}$

$$= \sigma^2 \times \frac{1}{T \Sigma_t (x_t - \bar{x})^2} \begin{bmatrix} \Sigma_t x_t^2 & -\Sigma_t x_t \\ -\Sigma_t x_t & T \end{bmatrix}$$

$$= \frac{\sigma^2}{\Sigma_t (x_t - \bar{x})^2} \begin{bmatrix} \Sigma_t x_t^2 / T & -\bar{x} \\ -\bar{x} & 1 \end{bmatrix} = \begin{bmatrix} Var(\hat{\beta}_1) & Cov(\hat{\beta}_1, \hat{\beta}_2) \\ Cov(\hat{\beta}_1, \hat{\beta}_2) & Var(\hat{\beta}_2) \end{bmatrix}$$

Ans 2-1

Assump that the electricity industry follows CRS (Constant Returns to Scale), then, Output increases by the same portional change as all inputs changes.
In this case, Cost represents all inputs.

$Cost = A \cdot Output^{\alpha}$, α must be 1 to satisfy assumption.
if $\alpha > 1$ Decreasing RS.
$\alpha < 1$ Increasing R.S.

i) $H_0 : \beta_2 = 1$ (CRS)
$H_1 : \beta_2 < 1$ (IRS)

ii) $$t = \frac{\hat{\beta}_2 - \beta_2}{s.e(\hat{\beta}_2)} = \frac{0.720 - 1}{0.0175} = -16 < -t_{(140, 5\%)} = -1.655$$

thus, we reject H_0

Ans 2-2

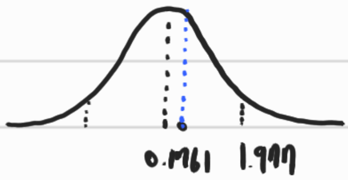
Where $y = A^{\alpha} B^{\beta} C^{\gamma}$, if function y is Homogeneous of Degree one in A, B, C
then, $\alpha + \beta + \gamma = 1$

i) $H_0 : \beta_3 + \beta_4 + \beta_5 = 1$
$H_1 : \beta_3 + \beta_4 + \beta_5 \neq 1$

ii) $t = \dfrac{\beta_3+\beta_4+\beta_5-1}{S.E(\hat\beta_3+\hat\beta_4+\hat\beta_5)}$, $Var(\hat\beta_3+\hat\beta_4+\hat\beta_5) = Var(\hat\beta_3)+Var(\hat\beta_4)+Var(\hat\beta_5)+2\left(Cov(\hat\beta_3,\hat\beta_4)+Cov(\hat\beta_3,\hat\beta_5)+Cov(\hat\beta_4,\hat\beta_5)\right)$

$= 0.1761 < t_{(140,\,2.5\%)} = 1.977$

then, Do not reject H_0



Ans 3-1 $\hat y_0 = \hat X_0'\hat\beta$ Denote 0 as the number of observations

first, we should find $\hat\beta$

i) $\hat\beta = (X'X)^{-1}X'y = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \hat\beta_1 \\ \hat\beta_2 \end{bmatrix}$

ii) $\hat\sigma^2 = \dfrac{1}{n-2}\sum \hat e_i^2 = \dfrac{1}{9}\sum(y_i-\hat y)^2 = \dfrac{1}{9}\sum(y_i-\hat\beta_1 x_{1i}-\hat\beta_2 x_{2i})^2$

$= \frac{1}{9}\sum(y_i-x_{1i})^2 = \frac{1}{9}\left\{\frac{7}{3}-4+2\right\} = \frac{1}{27}$

iii) $\hat y_{12} = X_{12}'\hat\beta = [5\ \ -2]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 5$

$se(\hat y_{12}) = \hat\sigma\sqrt{X_{12}'(X'X)^{-1}X_{12}+1} = \frac{1}{\sqrt{27}}\times\sqrt{27} = 1$

then, we can define 80% prediction intervals as

$\therefore\ \hat y_{12} \pm t_{(9,\,10\%)}\cdot S.E(\hat y_{12}) = 5 \pm 1.383\cdot 1 = [3.617,\ 6.383]$

iv) As same as for $\hat y_{13}$

$\hat y_{13} = 3$, $se(\hat y_{13}) = \frac{1}{\sqrt{27}}\times\sqrt{\frac{161}{3}} = 1.41$

$\therefore\ \hat y_{13} \pm t_{(9,\,10\%)}\cdot S.E(\hat y_{13}) = 3 \pm 1.383\cdot 1.41 = [3.05,\ 6.95]$

Ans 3-2 for the expected value of y_{12}, y_{13}, only change the variances.

μ_{12} is the mean of the line at X_{12}

i) μ_{12} has the same value of $\hat{y}_{12}$ $= 5$

ii) $s.e(\mu_{12}) = \hat{\sigma}\sqrt{X_{12}'(X'X)^{-1}X_{12}} = \frac{1}{\sqrt{27}} \times \sqrt{26} = 0.98$

80% prediction Intervals is then, μ_{12}

$$\mu_{12} \pm t_{(9,10\%)} \cdot s.e(\mu_{12}) = 5 \pm 1.383 \cdot 0.98 = [3.64, 6.36]$$

iii) As same as for μ_{13}

$$\mu_{13} \pm t_{(9,10\%)} \cdot s.e(\mu_{13}) = 3 \pm 1.383 \cdot 1.182 = [3.37, 6.63]$$

Ans 3-3 At Q 3-1, the Answer represents a particular point $\hat{y}_{12}, \hat{y}_{13}$

On the otherhands, the Answer of Q 3-2 predicts the mean of the line at X_{12}, X_{13}

Unlike the mean of the line, a particular point inevitably has extra noise term ϵ.

So, it brings more variance when we find prediction intervals

That's the reason why we got two different answer at the same circumstance.