[STA502] Problem set1_Q3

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Q.3) $y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + \varepsilon_i, \quad i = 1, 2, \cdots, n$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{11} \end{bmatrix} \quad X = \begin{bmatrix} X_{11} & X_{21} \\ X_{12} & X_{22} \\ \vdots & \vdots \\ X_{111} & X_{2n} \end{bmatrix} \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

→ intercept doesn't exist.

$$\hat{\beta} = (X'X)^{-1}X'y, \quad X = [x_1 \; x_2]$$

1) $$X'X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{111} \\ x_{21} & x_{22} & \cdots & x_{211} \end{bmatrix} \begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{21} \\ \vdots & \vdots \\ x_{111} & x_{211} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{11} x_{1i}^2 & \sum_{i=1}^{11} x_{1i} \cdot x_{2i} \\ \sum_{i=1}^{11} x_{2i} \cdot x_{1i} & \sum_{i=1}^{11} x_{2i}^2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1'x_1 & x_1'x_2 \\ x_2'x_1 & x_2'x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\therefore (X'X)^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

2) $$X'y = \begin{bmatrix} x_1'y \\ x_2'y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\therefore \hat{\beta} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

$$\therefore y_i = 1 \cdot x_{1i} + 0 \cdot x_{2i} + \varepsilon_i, \quad i = 1, 0, \cdots 11.$$

$$Var(\hat{\beta}) = Var((X'X)^{-1}X'y)$$
$$= (X'X)^{-1}X' Var(y)((X'X)^{-1}X')'$$
$$= (X'X)^{-1}X' Var(y) X (X'X)^{-1}$$

→ $Var(y) = Var(X\beta - \varepsilon) = Var(\varepsilon)$

$\therefore Var(y) = \sigma^2 I$

$$= \sigma^2 I (X'X)^{-1}X'X(X'X)^{-1}$$

$$\therefore Var(\hat{\beta}) = \sigma^2 (X'X)^{-1}$$

1-1) Sol. P.I. for observation 12.

$y_{12} = X_{12}'\beta + \varepsilon_{12}, \quad \hat{y}_{12} = X_{12}'\hat{\beta}$

$$Var(y_{12} - \hat{y}_{12}) = Var(y_{12}) + Var(\hat{y}_{12})$$
$$= Var(X_{12}'\beta + \varepsilon_{12}) + Var(X_{12}'\hat{\beta})$$
$$= Var(\varepsilon_{12}) + X_{12}' Var(\hat{\beta}) X_{12}$$
$$= \sigma^2 + \sigma^2 X_{12}'(X'X)^{-1}X_{12}$$
$$= \sigma^2(1 + X_{12}'(X'X)^{-1}X_{12})$$

i) $X_{12} = \begin{bmatrix} x_{112} \\ x_{212} \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \quad \therefore X_{12}' = [5 \; -2]$

$X_{12}'(X'X)^{-1}X_{12}$

$$= [5 \; -2]\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 5 \\ -2 \end{bmatrix} = 26$$

ii) σ^2 : we don't know it, instead,
we use s^2, which is $\frac{\sum_{i=1}^{n} \varepsilon_i^2}{n-p} \Rightarrow \frac{\sum(y_i - \hat{\beta}_1 x_{1i} - \hat{\beta}_2 x_{2i})}{n-p}$ (n : sample size, p : no. of parameters)

$$\therefore s^2 = \frac{\sum(y_i - 1 \cdot x_{1i} - 0 \cdot x_{2i})^2}{11-2} = \frac{\sum(y_i - x_{1i})^2}{9} = \frac{\sum(y_i^2 - 2 \cdot y_i x_{1i} + x_{1i}^2)}{9}$$

$$= \frac{\frac{7}{3} - 2 \cdot 2 + 2}{9} = \frac{1}{27}$$

$$\therefore Var(y_{12} - \hat{y}_{12}) = \frac{1}{27} \times (1 + 26) = 1.$$

prediction Interval for observation 12.

$$y_{12} = 1 \cdot x_{112} + 0 \cdot x_{212}$$
$$= 1 \times 5 + 0 \cdot (-2) = 5$$

$\therefore 5 \pm 1.383 \times \sqrt{1} \Rightarrow (3.617, 6.383)$ → $t_{0.1, 9} = 1.383$

1-2) Sol. P.I. for observation 13

i) $X_{13} = \begin{bmatrix} x_{113} \\ x_{213} \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix} \quad \therefore X_{13}' = [3 \; -7]$

$X_{13}'(X'X)^{-1}X_{13}$

$$= [3 \; -7]\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 3 \\ -7 \end{bmatrix} = \frac{158}{3}$$

$$Var(y_{13} - \hat{y}_{13}) = \sigma^2(1 + X_{13}'(X'X)^{-1}X_{13})$$
$$= \frac{1}{27} \times \left(1 + \frac{158}{3}\right) = \frac{1}{27} \times \left(\frac{161}{3}\right) = \frac{161}{81}$$

prediction Interval for observation 13.

$$y_{13} = 1 \cdot x_{113} + 0 \cdot x_{213}$$
$$= 1 \times 3 + 0 \cdot (-7) = 3$$

$$\therefore 3 \pm 1.383 \times \sqrt{\frac{161}{81}} = 3 \pm 1.383 \times 1.41 = 3 \pm 1.95003$$
$$\Rightarrow (1.04997, 4.95003)$$

2-1) Expected value $y_{12} \to \hat{y}_{12}$

$$\hat{y}_{12} = X_{12}'\hat{\beta}$$

$$Var(\hat{y}_{12}) = Var(X_{12}\hat{\beta})$$
$$= X_{12}' Var(\hat{\beta}) X_{12} \qquad \because Var(\hat{\beta}) = \sigma^2(X'X)^{-1}$$
$$= \sigma^2 X_{12}'(X'X)^{-1}X_{12}$$

from above, we know $\sigma^2 = \frac{1}{27}$, and $X_{12}'(X'X)^{-1}X_{12} = 26$.

$$\therefore Var(\hat{y}_{12}) = \frac{26}{27}$$

$$\hat{y}_{12} = X_{12}' \times \hat{\beta} = (5 \; -2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 5$$

prediction Intervals for Expected Value y_{12} is

$$5 \pm 1.383 \times \sqrt{\frac{26}{27}} = 5 \pm 1.383 \times 0.9813$$
$$\Rightarrow (3.642, 6.3571)$$

2-2) Expected value of $y_{13} \to \hat{y}_{13}$

$$\hat{y}_{13} = X_{13}'\hat{\beta}$$
$$Var(\hat{y}_{13}) = Var(X_{13}\hat{\beta})$$
$$= X_{13}' Var(\hat{\beta}) X_{13}$$
$$= \sigma^2 X_{13}'(X'X)^{-1}X_{13}$$

From above, we know $\sigma^2 = \frac{1}{27}$, and $X_{13}'(X'X)^{-1}X_{13} = \frac{158}{3}$

$$\therefore Var(\hat{y}_{13}) = \frac{158}{81}$$

$$\hat{y}_{13} = X_{13}'\hat{\beta} = [3 \; -7]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 3$$

prediction Intervals for Expected Value y_{13} is

$$3 \pm 1.383 \times \sqrt{\frac{158}{81}} = 3 \pm 1.383 \times 1.3966$$
$$\Rightarrow (1.0686, 4.9314)$$

3). The predicted intervals of observations of 12 & 13 is longer (wider) than the one for the expected value of y_{12} & y_{13}.
This can be interpreted that the differences are derived from that the predicted intervals of the observations 12, 13 take into consideration of ε, while the expected value of y_{12} & y_{13} don't.