Question 1-1

[Answer] $\min_\beta (y - X\beta)^2$, $\hat{\beta} = (X'X)^{-1}X'y$. Suppose that $X_i = [1 \quad x_i]$ as we have constant term in the equation.

$$X'X = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_T \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_T \end{bmatrix} = \begin{bmatrix} T & \sum_t x_t \\ \sum_t x_t & \sum_t {x_t}^2 \end{bmatrix} \tag{1}$$

$$(X'X)^{-1} = \frac{1}{\Delta} \begin{bmatrix} \sum_t {x_t}^2 & -\sum_t x_t \\ -\sum_t x_t & T \end{bmatrix} \tag{2}$$

$$\Delta = T \sum_t x_t^2 - (\sum_t x_t)^2 = T \sum_t (x_t - \overline{x})^2 \tag{3}$$

$$X'Y = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_T \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix} = \begin{bmatrix} \sum_t y_t \\ \sum_t x_t y_t \end{bmatrix} \tag{4}$$

$$\hat{\beta} = \begin{bmatrix} \hat{\beta_1} \\ \hat{\beta_2} \end{bmatrix} = (X'X)^{-1}X'Y = \frac{1}{\Delta} \begin{bmatrix} \sum_t {x_t}^2 & -\sum_t x_t \\ -\sum_t x_t & T \end{bmatrix} \begin{bmatrix} \sum_t y_t \\ \sum_t x_t y_t \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \sum_t {x_t}^2 \sum_t y_t - \sum_t x_t \sum_t x_t y_t \\ -\sum_t x_t \sum_t y_t + T \sum_t x_t y_t \end{bmatrix} \tag{5}$$

$$\hat{\beta_2} = \frac{-\sum_t x_t \sum_t y_t + T \sum_t x_t y_t}{T \sum_t (x_t - \overline{x})^2} = \frac{\frac{-\sum_t x_t}{T} \sum_t y_t + \sum_t x_t y_t}{\sum_t (x_t - \overline{x})^2} = \frac{\sum_t (x_t - \overline{x}) y_t}{\sum_t (x_t - \overline{x_t})^2} \tag{6}$$

$$\hat{\beta_1} = \frac{\sum_t {x_t}^2 \sum_t y_t - \sum_t x_t \sum_t x_t y_t}{T \sum_t (x_t - \overline{x})^2} = \frac{\frac{\sum_t y_t}{T} \sum_t x_t^2 - \sum_t y_t (\frac{\sum_t x_t}{T})^2 + \sum y_t (\frac{\sum_t x_t}{T})^2 - \frac{\sum_t x_t}{T} \sum_t x_t y_t}{\sum_t (x_t - \overline{x})^2} \tag{7}$$

$$\hat{\beta_1} = \frac{\frac{\overline{y}}{T}(T \sum_t x_t^2 - (\sum_t x_t)^2)}{\sum_t (x_t - \overline{x})^2} + \frac{\sum_t y_t (\overline{x})^2 - \overline{x} \sum_t x_t y_t}{\sum_t (x_t - \overline{x})^2} = \overline{y} - \frac{\overline{x} \sum_t (x_t - \overline{x}) y_t}{\sum_t (x_t - \overline{x})^2} \tag{8}$$

Question 1-2

[Answer] from $V(\hat{\beta}) = \sigma_\epsilon^2 (X'X)^{-1}$

$$V(\hat{\beta}) = \frac{\sigma_\epsilon^2}{\Delta} \begin{bmatrix} \sum_t {x_t}^2 & -\sum_t x_t \\ -\sum_t x_t & T \end{bmatrix} \tag{9}$$

Question 2-1

[Answer] The model can be represented as $y = \beta_1 + \beta_2 x_1 + \beta_3 x_2 + \beta_4 x_3 + \beta_5 x_4$ where $\beta_1 \sim t^*(-3.53, 1.77), \beta_2 \sim t^*(0.720, 0.0175), \beta_3 \sim t^*(0.436, 0.291), \beta_4 \sim t^*(0.220, 0.339), \beta_5 \sim t^*(0.427, 0.100)$ with $df = n - 2 = 145 - 143 = 142$. H_0: Electricity industry displays constant returns to scales means $\beta_2 = 1$. H_1: It displays increasing returns means $|\beta_2| > 1$. This test is an one-sided t test as the number of observations is small enough.

$$\tau = \frac{\hat{(\beta_2)} - 1}{0.0175} = \frac{(0.720 - 1)}{0.0175} = -16 \tag{10}$$

As above value is less than $t^*_{5\%}(142) = -1.6557$, we reject H_0.

Question 2-2

[Answer] I think that the related independent variables are x_3, x_4, and x_5 as they concerns prices (in the case of capital I think that it may be a kind of price) even though the variables use logarithmic function. Homogeneous of degree 1 means that if we replace x in $f(x)$ with λx, we can get $f(\lambda x) = \lambda f(x)$. $\beta log(\lambda x) = \beta(log(\lambda) + log(x))$. If we express this for all related $\beta's$ for homegeneous of degree one, $\beta_3 log(\lambda) + \beta_4 log(\lambda) + \beta_5 log(\lambda) = 0$. So $\beta_3 + \beta_4 + \beta_5 = 1$ (It seems like not correct!). Now H_0 is $\beta_3 + \beta_4 + \beta_5 = 1$ and H_1 is $\beta_3 + \beta_4 + \beta_5 \neq 1$ at 5% significance level. This is two sided t test. $|\tau| = \frac{\hat{\beta_3} + \hat{\beta_4} + \hat{\beta_5} - 1}{SE(\hat{\beta_3} + \hat{\beta_4} + \hat{\beta_5})}$. $SE(\hat{\beta_3} + \hat{\beta_4} + \hat{\beta_5}) = \sqrt{\sigma_{\hat{\beta_3}} + \sigma_{\hat{\beta_4}} + \sigma_{\hat{\beta_5}} + 2(\sigma_{\hat{\beta_{34}}} + \sigma_{\hat{\beta_{35}}} + \sigma_{\hat{\beta_{45}}})} = \sqrt{0.0847 + 0.115 + 0.0101 + 2(0.0237 + 0.0109 + 0.00663)} = 0.5406$. $|\tau| = |\frac{0.436 + 0.220 + 0.427 - 1}{0.5406}| = 0.083/0.5406 = 0.1535$. As this is not bigger than $t^*_{2.5\%}(142) = 1.9767$, we do not reject H_0.

Question 3-1

[Answer] $\min_\beta (y - X\beta)^2$, $\hat{\beta} = (X'X)^{-1}X'y$. Suppose that $X = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$

$$X = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \tag{11}$$

$$X' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} \tag{12}$$

$$X'X = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} x_1'x_1 & x_1'x_2 \\ x_2'x_1 & x_2'x_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{13}$$

$$(X'X)^{-1} = \frac{1}{2*2-1*1} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \tag{14}$$

$$X'Y = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} y = \begin{bmatrix} x_1'y \\ x_2'y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{15}$$

$$\hat{\beta} = (X'X)^{-1}X'Y = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{16}$$

Therefore, $\hat{\beta}_1 = 1$ and $\hat{\beta}_2 = 0$. The model is $y_i = x_{1i}$. As n is 11, this follows t distribution with degree of 9. 80% prediction interals for the dependent variable y is the following equation from out 1st lecture note:

$$X \pm t_{\alpha/2} S \sqrt{\frac{1}{n} + \frac{(x^* - \overline{x})}{\sum_i (x_{1i} - \overline{x_1})^2} + 1} \tag{17}$$

Here $t_{0.1}(9) = 1.383$, $S = \hat{\sigma}$, $\overline{x} = 0$ (as the model is fitted as $y = x_1$ (still question!!)). For $\hat{\sigma}$ we can calculate it from the following:

$$X \pm t_{\alpha/2} S \sqrt{\frac{1}{n} + \frac{(x^* - \overline{x})}{\sum_i (x_{1i} - \overline{x_1})^2} + 1} = X \pm t_{\alpha/2} S \sqrt{\frac{1}{n} + \frac{x^*}{\sum_i (x_{1i})^2} + 1} \tag{18}$$

$$\hat{\sigma}^2 = \frac{1}{n-2} \sum_i \hat{\epsilon_i}^2 = \frac{1}{n-2} \sum_i (y_i - x_{1i})^2 = \frac{1}{n-2} (\sum_i {y_i}^2 - 2 \sum_i y_i x_{1i} + \sum_i x_{1i}^2) = \frac{1}{9}(\frac{7}{3} - 4 + 2) = \frac{1}{27} \tag{19}$$

Finally, at $(x_1, x_2) = (5, -2)$,

$$5 \pm (1.383)(\sqrt{\frac{1}{27}}) \sqrt{\frac{1}{11} + \frac{5}{(x_1'x) = 2} + 1} = 5 \pm 0.504 = [4.496, 5.504] \tag{20}$$

and at $(x_1, x_2) = (3, -7)$,

$$3 \pm (1.383)(\sqrt{\frac{1}{27}}) \sqrt{\frac{1}{11} + \frac{3}{(x_1'x) = 2} + 1} = 3 \pm 0.504 = [2.496, 3.504] \tag{21}$$

Question 3-2

[Answer] Different from 3-1, this considers its mean. As we have less variance, we have the following equation:

$$X \pm t_{\alpha/2} S \sqrt{\frac{1}{n} + \frac{(x^* - \overline{x})}{\sum_i (x_{1i} - \overline{x_1})^2}} \tag{22}$$

In the case of $(x_1, x_2) = (5, -2)$

$$5 \pm (1.383)(\sqrt{\frac{1}{27}}) \sqrt{\frac{1}{11} + \frac{5}{(x_1'x) = 2}} = 5 \pm 0.428 = [4.572, 5.428] \tag{23}$$

In the case of $(x_1, x_2) = (3, -7)$

$$3 \pm (1.383)(\sqrt{\frac{1}{27}}) \sqrt{\frac{1}{11} + \frac{3}{(x_1'x) = 2}} = 3 \pm 0.428 = [2.572, 3.428] \tag{24}$$

Question 3-3

[Answer] Yes, they are different. Estimating a precise point from model has more uncertainty as compared to estimating its mean. Therefore for estimating a precise point from the model we add additional +1 while there is not for mean as you can see the equations of 18 and 22 So the range of mean at the point has narrower prediction intervals.