Problem 3

[Answer]

Q3) $y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + \varepsilon_i \quad \varepsilon_i \sim \text{iid } N(0, \sigma_\varepsilon^2)$

Denote $X = [x_1 \quad x_2]$,

$$Y = X\beta + \varepsilon$$

$$\hat{\beta} = \underbrace{\begin{bmatrix} x_1'x_1 & x_1'x_2 \\ x_2'x_1 & x_2'x_2 \end{bmatrix}^{-1}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} Y}_{B},$$

$$A = \frac{1}{x_1'x_1 \cdot x_2'x_2 - x_1'x_2 \cdot x_2'x_1} \begin{bmatrix} x_2'x_2 & -x_1'x_2 \\ -x_2'x_1 & x_1'x_1 \end{bmatrix}$$

$$= \frac{1}{2 \cdot 2 - 1 \cdot 1} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} x_1'y \\ x_2'y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

from A and B, we have $\hat{\beta} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Denote $Y_p = \begin{bmatrix} y_{12} \\ y_{13} \end{bmatrix}$, $X_p = \begin{bmatrix} 5 & -2 \\ 3 & -7 \end{bmatrix}$

(1) Here, we wish to predict Y_p associated with X_p. The actual equation should be:

$$Y_p = X_p' \beta + \varepsilon_p$$

Suppose $\varepsilon_p \sim \text{iid } N(0, \sigma_\varepsilon^2)$, it follow GM,

$$\hat{Y}_p = X_p' \hat{\beta} \quad (\text{BLUE of } E[Y_p | X_p] = X_p' \beta)$$

$$\hat{Y}_p = X_p \cdot (X'X)^{-1} X'Y$$

Prediction error of this estimator would be

$$e_p = \hat{Y}_p - Y_p = (\hat{\beta} - \beta)' X_p - \varepsilon_p$$

$$Var(e_p | X, X_p) = Var[(\hat{\beta} - \beta)' X_p | X, X_p] + Var(\varepsilon_p)$$
$$= X_p' Var(\hat{\beta} | X_p, X) X_p + \sigma_\varepsilon^2$$
$$= \sigma_\varepsilon^2 + X_p' \cdot [\sigma_\varepsilon^2 (X'X)^{-1}] X_p$$

$$\sigma_\varepsilon^2 = (N-k)^{-1} \cdot \varepsilon_p' \varepsilon_p = (11-2)^{-1} \cdot \langle y'y - \hat{\beta}' X' X \hat{\beta} \rangle$$
$$= (1/9) \cdot \left\langle \frac{7}{3} - [1 \ 0] \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = \frac{7}{27} - \frac{2}{9} = \frac{1}{27}$$

$$\therefore Var(e_{p12} | X, X_{p12}) = \frac{1}{27} + \frac{1}{27} \cdot \frac{1}{3} \cdot [5 \ -2] \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \end{bmatrix} = 1 \cdots ①$$

$$Var(e_{p13} | X, X_{p13}) = \frac{1}{27} + \frac{1}{27} \cdot \frac{1}{3} [3 \ -7] \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -7 \end{bmatrix} = \frac{1}{27}\left(1 + \frac{158}{3}\right) \cdots ②$$

(cont.)
we now could derive the interval at $\alpha = 0.2$

$$P\left[-t_{\alpha/2}[N-K] < \frac{\hat{y}_{pi} - y_{pi}}{Var(\hat{y}_{pi} - y_{pi} | X)} < t_{\alpha/2}[n-k]\right] = 1 - \alpha$$

Therefore, 80% prediction interval for y_{12}, y_{13}:

For y_{12}, $5 \pm 1.383 \sqrt{e_{p12}}$, $\therefore (3.617, 6.383)$

For y_{13}, $3 \pm 1.383 \sqrt{e_{p13}}$, $\therefore (1.050, 4.949)$

where $e_{p12} = ①$, $e_{p13} = ②$.

(2) We would derive PI for $E[y_{12} | X]$, $E[y_{13} | X]$

Denote $e_{\bar{p}} = \hat{Y}_p - E[y_{12} | X]$

$$Var(e_{\bar{p}}) = Var(\hat{Y}_p - E[y_{12} | X])$$
$$= Var(\hat{Y}_p) \quad \because E[\sigma_\varepsilon^2] = 0, \text{ by definition}$$

from ①, ②, we have

$$Var(e_{\bar{p}12}) = \frac{26}{27}, \quad Var(e_{\bar{p}13}) = \frac{158}{81}$$

Therefore, 80% prediction interval for $E[y_{12} | X]$ and $E[y_{13} | X]$:

For $E[y_{12} | X]$, $5 \pm 1.383 \cdot \sqrt{\frac{26}{27}}$, $(3.642, 6.357)$

For $E[y_{13} | X]$, $3 \pm 1.383 \sqrt{\frac{158}{81}}$, $(1.068, 4.931)$

(3) Prediction intervals for y_{12} and y_{13} are wider than those for $E[y_{12} | X]$, $E[y_{13} | X]$. This is because the variances for y_{12}, y_{13}