Question 1. Let $gGDP_t$ denote the annual percentage change in gross domestic product and let int_i denote a short-term interest rate. Suppose that $gGDP_t$ is related to interest rate by

$$gGDP_t = \alpha_0 + \delta_0 int_t + \delta_1 int_{t-1} + u_t$$

where u_t is uncorrelated with int_t, int_{t-1}, and all other past values of interest rates.
Suppose that the Federal Reserve follows the policy rule:

$$int_t = \gamma_0 + \gamma_1(gGDP_{t-1} - 3) + v_t,$$

where $\gamma_1 > 0$. In behavioral terms, $\gamma_1 > 0$ means that when last year's GDP growth is above 3%, the Fed increases interest rates to prevent an "overheated" economy.

(a) If v_t is uncorrelated with all past values of int_t and u_t, argue that int_t must be correlated with u_{t-1}. (Hint: Lag the first equation for one time period and substitute for $gGDP_{t-1}$ in the second equation.) We want to show that $\mathrm{Cov}(int_t, u_{t-1}) \neq 0$

$$gGDP_{t-1} = \alpha_0 + \delta_0 int_{t-1} + \delta_1 int_{t-2} + u_{t-1}$$

$$\begin{aligned} int_t &= \gamma_0 + \gamma_1(\alpha_0 + \delta_0 int_{t-1} + \delta_1 int_{t-2} + u_{t-1} - 3) + v_t \\ &= (\gamma_0 + (\alpha_0 - 3)\gamma_1) + \gamma_1\delta_0 int_{t-1} + \gamma_1\delta_1 int_{t-2} + \gamma_1 u_{t-1} + v_t \end{aligned}$$

$$\mathrm{Corr}(int_t, u_{t-1}) = \gamma_1 > 0$$

$$\therefore \mathrm{Cov}(int_t, u_{t-1}) \neq 0$$

(b) Which Gauss-Markov assumption does $\mathrm{Corr}(int_t, u_{t-1})$ violate?

$$\mathrm{Cov}(int_t, u_{t-1}) = E(int_t u_{t-1}) - E(int_t)E(u_{t-1}) = E(int_t u_{t-1}) \quad \because E(u_t) = 0$$

$$int_t u_{t-1} = (\gamma_0 + (\alpha_0 - 3)\gamma_1)u_{t-1} + \gamma_1\delta_0 int_{t-1}u_{t-1} + \gamma_1\delta_1 int_{t-2}u_{t-1} + \gamma_1 u_{t-1}^2 + v_t u_{t-1}$$

$$\begin{aligned} E(int_t u_{t-1}) &= (\gamma_0 + (\alpha_0 - 3)\gamma_1)E(u_{t-1}) + \gamma_1\delta_0 E(int_{t-1})E(u_{t-1}) + \gamma_1\delta_1 E(int_{t-2})E(u_{t-1}) + \gamma_1 E(u_{t-1}^2) + E(v_t u_{t-1}) \\ &\because u_t \ \& \ int_t, int_{t-1} \text{are independent} \\ &= \gamma_1 \mathrm{Var}(u_{t-1}) + E(v_t u_{t-1}) \end{aligned}$$

$$\mathrm{Corr}(int_t, u_{t-1}) = \frac{\gamma_1 \mathrm{Var}(u_{t-1}) + E(v_t u_{t-1})}{\sqrt{\mathrm{Var}(v_t)\mathrm{Var}(u_{t-1})}} = \gamma_1$$

$$\mathrm{Corr}(v_t u_{t-1}) = \gamma_1 \left(1 - \sqrt{\frac{\mathrm{Var}(u_{t-1})}{\mathrm{Var}(v_t)}}\right)$$

If $\mathrm{Var}(u_{t-1}) = \mathrm{Var}(v_t)$, then $\mathrm{Corr}(v_t u_{t-1}) = 0$. However, In other cases, correlations between error terms occur. It means this time series model can have autocorrelation.